Approximation of Measures by Measures supported on Curves

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The approximation of probability measures on manifolds by measures supported in lower dimensions is a classical task in approximation and complexity theory with a wide range of applications. In this talk, we focus on measures supported on curves, where we highlight two approaches:

i) *Principal curves* are natural generalizations of principal lines arising as first principal components in the Principal Component Analysis. They can be characterized from a stochastic point of view as so-called self-consistent curves based on the conditional expectation and from the variational-calculus point of view as saddle points of the expected difference of a random variable and its projection onto some curve, where the current curve acts as argument of the energy functional. We show that principal curves in $\mathbb{R}^d$ can be computed as solutions of a system of ordinary differential equations and we provide several examples for principal curves related to the uniform distribution on certain domains, see [1].

ii) *Discrepancy minimizing curves* aim to minimize so-called discrepancies between measures. Besides proving optimal approximation rates in terms of the curve's length and Lipschitz constant, we are interested in the numerical minimization of the discrepancy between a given probability measure and the set of push-forward measures of Lebesgue measures on the unit interval by Lipschitz curves. We present numerical examples for measures on the 3-dimensional torus, the 2-sphere, the rotation group on $\mathbb{R}^3$ and the Grassmannian of all 2-dimensional linear subspaces of $\mathbb{R}^3$. Our algorithm of choice is a conjugate gradient method on these manifolds, which incorporates second-order information. For efficient gradient and Hessian evaluations within the algorithm, we approximate the given measures by truncated Fourier series and use fast Fourier transform techniques on these manifolds, see [2]. Finally, we are interested in the relation of our approach to Wasserstein gradient flows of discrepancies.

Joint work with: R. Beinert, A, Bërdëllima, M. Ehler, M. Gräf, S. Neumayer

References

[1] R. Beinert, A, Bërdëllima, M. Gräf, and G. Steidl. On the dynamical system of principal curves in $\mathbb{R}^d$. *Preprint arXiv*:2108.00227, 2021.

[2] M. Ehler, M. Gräf, S. Neumayer, and G. Steidl. Curve based approximation of measures on manifolds by discrepancy minimization. *Foundations of Computational Mathematics*, 21(6), 1595–1642, 2021.