Lattices enumeration via linear programming.

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Given a positive integer d and $\boldsymbol{a}^{(1)}, \dots, \boldsymbol{a}^{(m)}$ m vectors in $\mathbb{R}^d$, $\{k_1\boldsymbol{a}^{(1)} + \cdots + k_m\boldsymbol{a}^{(m)} : k_1, \dots, k_m \in \mathbb{Z}\} \subset \mathbb{R}^d$ is the so-called lattice generated by the family of vectors, or by the matrix $\boldsymbol{A} = (\boldsymbol{a}^{(1)}|\dots|\boldsymbol{a}^{(m)}) \in \mathbb{R}^{d\times m}$. In high dimensional integration, prescribed lattices are used for constructing reliable quadrature schemes. The quadrature points are the lattice points lying on the integration domain, typically the unit hypercube $[0, 1)^d$ or a shifted hypercube. It is crucial to be able to enumerate the lattice points in such domains inexpensively. Undeniably, the lack of fast enumeration procedures hinders the applicability of lattice rules. Existing enumeration procedures exploit intrinsic properties of the lattice at hand, such as $\mathbb{Z}$-periodicity, orthogonality, recurrences, etc, e.g. [1, 2, 3, 4]. We present a general-purpose lattice enumeration strategies based on linear programming, [5]. We demonstrate how to combine duality and parametric linear programming in order to accelerate these strategies, producing performances comparable to the enumeration strategies that are fine-tuned to special lattices. In addition, we discuss a variety of relaxation and reduction techniques that allow further acceleration of the introduced algorithms. Numerical experiments in high dimension are also presented.

References

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