Wasserstein distance, the Witten Laplacian, and Applications

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This talk considers the problem of computing a linear approximation of quadratic Wasserstein distance W_2. In particular, we compute an approximation of the negative homogeneous weighted Sobolev norm whose connection to Wasserstein distance follows from a classic linearization of a general Monge-Ampére equation. We reduce the computational problem to solving an elliptic boundary value problem involving the Witten Laplacian, which is a Schrödinger operator of the form

$$H = -\Delta + V,$$

where V is a potential that depends on f, see Figure 1. We show that this connection provides a method of computation whose computational cost can be controlled by the amount of regularization used when defining the potential. For the case of probability distributions on the unit square $[0, 1]^2$ represented by $n \times n$ arrays we present a fast code and several numerical examples demonstrating this approach.

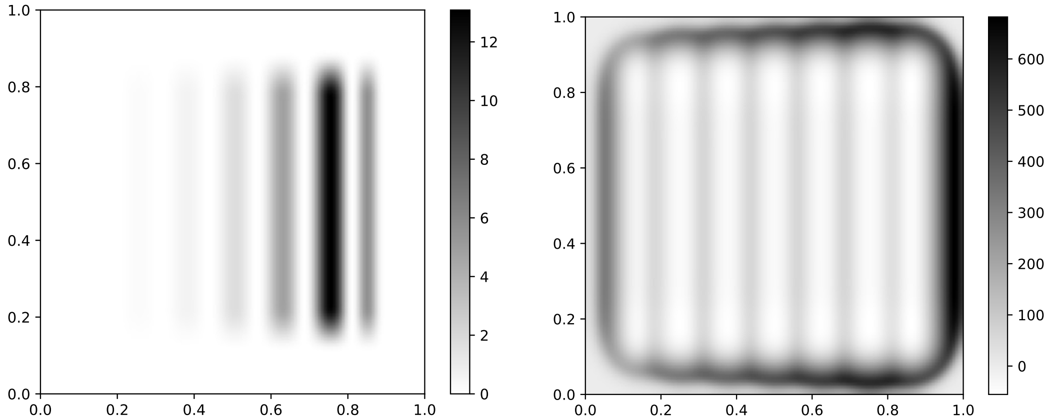

Figure 1: An example of a function f (left) and its regularized potential V (right).

The connection between the weighted negative homogeneous Soboelv norm and the Witten Laplacian has a number of interesting applications; in particular, we discuss applications to defining an embedding and smoothing. First, given probability density functions f, g, h we define an embedding $g \mapsto \Phi_f(g)$ such that

$$\|\Phi_f(g) - \Phi_f(h)\|_{L^2} \approx W_2(g, h),$$

whenever g and h are close to f (in a precise sense). Second, we consider the problem of smoothing g with respect to a diffusion defined by f. In particular, we define the diffusion operator

$$g \mapsto e^{-\tau H} g,$$

where H is the Witten Laplacian whose potential depends on f, and $\tau > 0$ is the diffusion time parameter that controls the amount of smoothing. Further applications and connections to other methods will also be discussed.

Joint work with: Philip Greengard, Jeremy G. Hoskins, Amit Singer

References

[1] Philip Greengard, Jeremy G. Hoskins, Nicholas F. Marshall, and Amit Singer, *On a linearization of quadratic wasserstein distance*, arXiv e-print (2022).