Direct inversion methods for the multivariate nonequispaced fast Fourier transform

Melanie Kircheis
Faculty of Mathematics
Chemnitz University of Technology
melanie.kircheis@math.tu-chemnitz.de

The well-known discrete Fourier transform (DFT) can easily be generalized to arbitrary nodes in the spatial domain. For $M \in 2\mathbb{N}$ we define the multi-index set

$$\mathcal{I}_{\boldsymbol{M}} \coloneqq \mathbb{Z}^d \cap \left[-\tfrac{M}{2}, \tfrac{M}{2}\right)^d = \left\{\boldsymbol{k} \coloneqq (k_t)_{t=1}^d \in \mathbb{Z}^d \colon\ -\tfrac{M}{2} \leq k_t < \tfrac{M}{2},\ t = 1, \ldots, d\right\}$$

with cardinality $|\mathcal{I}_{\boldsymbol{M}}| = M^d$. For given coefficients $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$, as well as arbitrary nodes $\boldsymbol{x}_j \in \left[-\frac{1}{2}, \frac{1}{2}\right)^d$, $j = 1, \ldots, N$, in the space domain, we are interested in the fast evaluation of the N values

$$f(\boldsymbol{x}_j) = \sum_{\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{x}_j}\,, \quad j = 1, \ldots, N\,.$$

The fast procedure for this generalization is referred to as nonequispaced fast Fourier transform (NFFT) and has complexity $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|) + N)$.

Various applications such as MRI, solution of PDEs, etc. are interested in the inverse problem, i. e., computing Fourier coefficients $\hat{f}_{\boldsymbol{k}} \in \mathbb{C}$, $\boldsymbol{k} \in \mathcal{I}_{\boldsymbol{M}}$, from given nonequispaced data $f_j = f(\boldsymbol{x}_j) \in \mathbb{C}$, $j = 1, \ldots, N$. In contrast to iterative solvers we study direct methods for this inversion, when we are in the overdetermined setting. For this purpose, we use the matrix representation of the NFFT. Besides the study of the approach of so-called density compensation factors, we introduce a new method using optimization. Modifying one of the matrix factors of the NFFT leads to an optimization problem, which can simply be solved in a precomputational step using normal equations. Thereby, we are able to compute an inverse NFFT up to a certain accuracy by means of a modified adjoint NFFT, which preserves the arithmetic complexity of $\mathcal{O}(|\mathcal{I}_{\boldsymbol{M}}| \log(|\mathcal{I}_{\boldsymbol{M}}|) + N)$.

Joint work with: Daniel Potts.

References

[1] M. Kircheis and D. Potts. Efficient multivariate inversion of the nonequispaced fast Fourier transform. PAMM, 20(1):e202000120, 2021.