Robust Eigenvectors of Symmetric Tensors

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With the rising demand for techniques to handle massive, high-dimensional datasets, many scientists have turned to finding adaptations of matrix algorithms to high-order arrays, known as tensors. In particular, the notions of eigenvalues and singular values can be generalized to the tensor case, which are particularly important due to their link to the rank-one approximation problem. The main obstacle, however, is that computing these quantities can be an NP-hard problem [1] for a general tensors. However, for a special case of orthogonal tensors, eigenvectors are well understood. In this talk, we report new results on eigenvectors of symmetric tensors and convergence of tensor power method for non-orthogonal case [4].

A real order d tensor $\mathcal{T} \in \mathbb{R}^{n\times\cdots\times n}$ is said to be *symmetric* if for all permutations of indices $\mathcal{T}_{i_1,\ldots,i_d} = \mathcal{T}_{i_{\sigma(1)},\ldots,i_{\sigma(d)}}$. A vector $\mathbf{v} \in \mathbb{R}^n$ is an *eigenvector* of $\mathcal{T}$ with *eigenvalue* $\mu \in \mathbb{R}$ if

$$\mathcal{T} \cdot \mathbf{v}^{d-1} = \mu \mathbf{v},$$

where $\mathcal{T} \cdot \mathbf{v}^{d-1}$ is a vector defined by *contracting* $\mathcal{T}$ by $\mathbf{v}$ along all of its modes except for one. The eigenvectors of $\mathcal{T}$ are the fixed points (up to sign) of an iterated method called the *tensor power method* given by

$$\mathbf{x}_{k+1} \mapsto \frac{\mathcal{T} \cdot \mathbf{x}_k^{d-1}}{\|\mathcal{T} \cdot \mathbf{x}_k^{d-1}\|}.$$

Yet another important characterization of the eigenvectors is that [2] they are the critical points of the symmetric best rank-one approximation problem $\min_{c,\mathbf{v}} \|\mathcal{T} - c\mathbf{v}^{\otimes d}\|_F^2$.

A non-symmetric version ot the tensor power method for non-symmetric tensors is known to be globally convergent [5]. For the symmetric case, fewer results are available on the convergence, examples are known when the method does not converge at all [2]. Our main result gives a sufficient condition on an eigenvector to be robust (i.e., to be an attracting fixed point of the power iteration).

Theorem 1. *let $\mathcal{T}_d$ be a symmetric $n \times \cdots \times n$ order-d tensor with symmetric decomposition*

$$\mathcal{T}_d = \sum_{i=1}^{r} \lambda_i \mathbf{v}_i^{\otimes d}, \tag{1}$$

with $\|\mathbf{v}_i\| = 1$ for all i. Then there exists a $D \in \mathbb{N}$ such that for all $d \geq D$, if $\mathbf{v}_j$ is an eigenvector of $\mathcal{T}_d$ with non-zero eigenvalue, then $\mathbf{v}_j$ is a robust eigenvector of $\mathcal{T}_d$.

Theorem 1 allows us to obtain find eigenvectors for some classes of non-orthogonal tensors, where the vectors $\mathbf{v}_j$ belong to an *equiangular set* or an *equiangular tight frame*, see [3] for more details.

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References

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