Tikhonov Regularization of Circle-Valued Signals

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It is common to have to process signals or images whose values are cyclic and can be represented as points on the complex circle, like wrapped phases, angles, orientations, or color hues. We consider a Tikhonov-type regularization model to smoothen or interpolate circle-valued signals defined on arbitrary graphs. We propose a convex relaxation of this nonconvex problem as a semidefinite program, and an efficient algorithm to solve it.

Let $\mathbb{S} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ denote the complex unit circle. We want to estimate a signal $x = (x_n)_{n\in V}$, with values $x_n \in \mathbb{S}$, defined on an undirected graph (V, E), where V is the set of nodes and E is the set of edges, which are sets of two distinct nodes. Typically, we are given a noisy signal $y = (y_n)_{n\in V}$ defined on the same graph and the sought signal x is a smoothed, or denoised, version of y, which achieves a tradeoff between closeness to y and smoothness, in some sense.

For real-valued signals, Tikhonov-regularized smoothing consists in solving the following convex problem: given $y = (y_n)_{n\in V}$ and nonnegative weights $(w_n)_{n\in V}$ and $(\lambda_{n,n'})_{\{n,n'\}\in E}$, $x = (x_n)_{n\in V}$ is the solution to

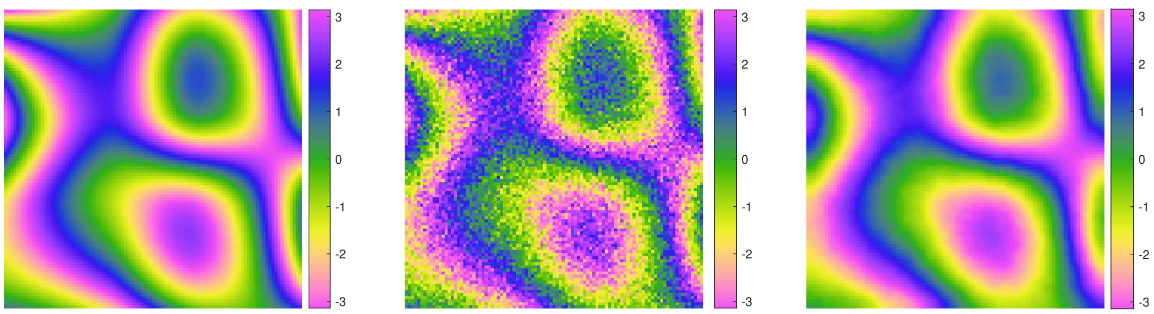
$$\underset{x_n\in\mathbb{R}\,:\,n\in V}{\text{minimize}} \sum_{n\in V} \frac{w_n}{2}(x_n - y_n)^2 + \sum_{\{n,n'\}\in E} \frac{\lambda_{n,n'}}{2}(x_n - x_{n'})^2. \tag{1}$$

We formulate a similar problem for signals x and y with values in $\mathbb{S}$. For this, we adopt a Bayesian view and replace the Gaussian distribution, whose anti-log-likelihood gives the squared differences in (1), by the von Mises distribution. This yields the nonconvex problem

$$\underset{x_n\in\mathbb{S}\,:\,n\in V}{\text{minimize}} \sum_{n\in V} w_n\big(1 - \Re(x_n y_n^*)\big) + \sum_{\{n,n'\}\in E} \lambda_{n,n'}\big(1 - \Re(x_n x_{n'}^*)\big), \tag{2}$$

where $\Re$ denotes the real part and $\cdot^*$ denotes the complex conjugation. The main contribution of this work is a new convex relaxation of this nonconvex problem, which takes the form of linear minimization over the product of complex elliptopes. It originates from a fomulation using optimal transport of measures on the circle, parameterized by a finite number of their Fourier coefficients [2]. The benefits of the proposed approach will be illustrated with numerical experiments, like the one in Figure 1. This work is described in [1].

[1] L. Condat, "Tikhonov Regularization of Circle-Valued Signals," preprint arXiv:2108.02602, 2021.
[2] L. Condat, "Atomic norm minimization for decomposition into complex exponentials and optimal transport in Fourier domain,' Journal of Approximation Theory, vol. 258, Oct. 2020.



(a) (b) (c)

Figure 1: Denoising of a circle-valued image: (a) the ground-truth image, (b) the noisy image, (c) the image denoised with the proposed method, for which the convex relaxation is exact: the image is the exact solution of the nonconvex problem. All images have their values in $\mathbb{S}$, whose argument in $(-\pi, \pi]$ is displayed using a cyclic colormap.