Phase retrieval of entire functions

Matthias Wellershoff
Seminar for Applied Mathematics, ETH Zurich
matthias.wellershoff@math.ethz.ch

In this talk, we will discuss what information about an entire function may be recovered from measurements of its magnitude on different subsets of the complex plane. This *phase retrieval problem for entire functions* plays a decisive role in many contemporary works on the recovery of signals from magnitude-only measurements [1, 2, 4, 5]. We will present the theory for the phase retrieval of entire functions starting with some basic observations. We will then discuss Mc Donald's characterisation of entire functions whose magnitudes agree on a single line in the complex plane [6] and, finally, propose some extensions to situations in which magnitude information on two or more lines is available [7].

Our presentation will be based on well-known complex-analytic techniques: the Hadamard factorisation theorem, in particular, plays an important role in the characterisation of entire functions whose magnitudes agree on certain lines in the complex plane. The main new results which we will discuss are

1. a full characterisation of all (finite order) entire functions whose magnitudes agree on two arbitrary lines in the complex plane — this extends work by Jaming [4] — and
2. a full characterisation of all entire functions of exponential-type whose magnitudes agree on infinitely many equidistant parallel lines.

These new results have interesting applications to Gabor phase retrieval. Among other things, they allow for a full characterisation of all L^2-functions whose Gabor transform magnitudes agree on two arbitrary lines in the time-frequency plane. We also emphasise that our theory can be used to generate functions in L^2 which do not agree up to global phase but whose magnitudes agree on infinitely many equidistant parallel lines. We have recently used this insight to find counterexamples for sampled Gabor phase retrieval [1], i.e. functions in L^2 which do not agree up to global phase but whose Gabor transform magnitudes agree on fairly general lattices (this work has just been generalised in [3]).

Finally, a good understanding of the theory of functions allows for the construction of many more interesting examples. Personal favourites of ours are the "universal counterexamples" for sampled Gabor phase retrieval: those are functions $f \in L^2(\mathbb{R})$ and $(g_n)_{n\geq 1} \in L^2(\mathbb{R})$ such that for all $n \geq 1$, it holds that f and g_n do not agree up to global phase while their Gabor transform magnitudes agree on the semidiscrete set $\mathbb{R} \times \frac{1}{n}\mathbb{Z}$ [7].

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References

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