Convexification of branched transport

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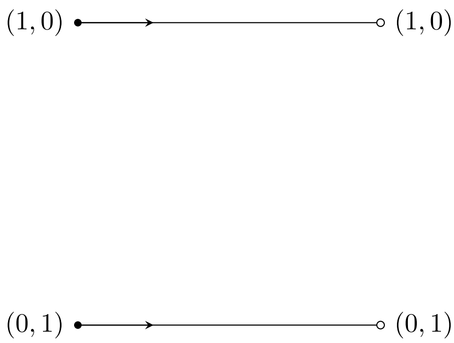
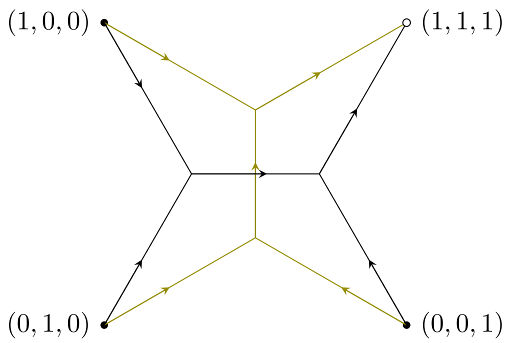
The *branched transport problem* is a non-convex and non-smooth variational problem on Radon measures. In this one tries to find an optimal mass flux $\mathcal{F}$ (in the Eulerian formulation a vector-valued measure) between two given probability measures μ_+ and μ_-, which may for example describe the initial and final distribution of some material. The optimality here is with respect to a subadditive *transportation cost* describing the effort $\tau(m)$ to move an amount of material m per unit distance. The subadditivity of the transportation cost leads to a branched structure of the network on which the transport described by $\mathcal{F}$ takes place. We study a convexification of this problem to a *multimaterial transport problem* [1], which is also of its own interest. The cost function h appearing in this relaxed problem is nonnegative homogeneous and describes the effort to move combinations of certain artificially generated materials. With regard to the branched transport problem, for some cases one can design the function h such that the relaxation is tight.





One can for example use the multimaterial problem to model the Steiner problem of finding the network with minimal total length connecting four nodes positioned at the corners of a square. In branched transport one would use $\tau = 1_{(0,\infty)}$. Using the approach with two materials in the left picture, one does not obtain the right solution to the Steiner problem (the network is not connected). The multimaterial distributions are indicated through the vectors and marks at the nodes, $\vec{\mu}_+ = (\delta_{(-\ell,\ell)}, \delta_{(-\ell,-\ell)})$ and $\vec{\mu}_- = (\delta_{(\ell,\ell)}, \delta_{(\ell,-\ell)})$. The approach with three materials in the right picture is the better choice: marking one point as the sink yields all possible solutions. Alternatively, one can swap the entries of $\vec{\mu}_-$ in the left figure.
Using Fenchel's duality theorem in combination with a result on the conjugation of integral functionals [2], we view the multimaterial transport problem as the dual problem to a variant of the Kantorovich–Rubinstein formula for the Wasserstein distance (which in its classical form is used to solve the transportation problem with $\tau(m) = m$). The primal-dual optimality conditions then naturally lead to our definition of a calibration, a certificate for optimality of a minimizing candidate of the (dual) multimaterial transport problem. Further, we relax the function space of the primal problem to ensure existence of solutions and simultaneously derive the notion of a weak calibration. We give conditions under which a weak calibration can be represented by a (strong) calibration and provide a procedure of how to construct a calibration from a weak calibration. Further, we give examples of calibrations and will use them to prove properties of branched transport networks.

Joint work with: Bernhard Schmitzer, Benedikt Wirth.

References

[1] A. Marchese, A. Massaccesi, R. Tione. A multi-material transport problem and its convex relaxation via rectifiable G-currents. *SIAM Journal on Mathematical Analysis*, 51(3):1965–1998, 2019.

[2] M. Valadier. Convex integrands on Suslin locally convex spaces. *Pacific Journal of Mathematics*, 20(3):267–276, 1975.