Super-resolution of generalized spikes and spectra of confluent Vandermonde matrices

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The problem of computional super-resolution (SR) is to recover the fine details of an unknown object from inaccurate measurements of inherently low resolution (see [1]). In recent years, there is much intrest in the problem of reconstructing a signal modeled by a linear combination of Dirac $\delta-$distributions with higher-order derivatives:

$$\mu(x)=\sum_{j=1}^{s}\sum_{l=0}^{n-1}a_{j,l}\delta^{(l)}(x-\xi_j),\ \xi_j\in[-\pi,\pi) \tag{1}$$

from noisy Fourier measurements:

$$y_k:=\hat{\mu}(k)+\eta_k,\quad k=0,1,...,M,\ |\eta_k|\le\varepsilon, \tag{2}$$

where $\delta^{(l)}$ is the l-th distributional derivative of the Dirac measure. The stability of this generalized problem is of importance in several applications including modern sampling theory beyond the Nyquist rate, algebraic signal recovery, and multi-exponential analysis, to name a few (see [4] and references therein).

For $n=1$, the measurment vector $\boldsymbol{y}=\{y_k\}_{k=0}^{M}\in\mathbf{C}^{M+1}$ can be expressed as $\boldsymbol{y}=V\boldsymbol{a}+\boldsymbol{\eta}$, where V is the $(M+1)\times s$ Vandermonde matrix with the nodes on the unit circle:

$$V:=[e^{ik\xi_j}]_{k=0,\dots,M}^{j=1,\dots,s}.$$

In order to describe the stability of this inverse problem, suppose that the nodes ξ_j belong to a grid of step size Δ, and define the super-resolution factor (SRF) as $\frac{1}{(M\Delta)}$. Suppose that at most $\ell\le s$ nodes form a "cluster" of size $O(\Delta)$. In the "super-resolution regime" $SRF\gg 1$ [3, 2] showed that $\sigma_{min}(V)$ scales like $SRF^{\ell-1}$ and consequently the worst-case reconstruction error rate of μ as in (1) from noisy measurements (2) is of the order $SRF^{2\ell-1}\varepsilon$ and it is minimax, meaning that on one hand, it is attained by a certain algorithm for all signals of interest, and on the other hand, there exist worst case examples for which no algorithm can achieve an essentially smaller error.

In this work we extend the above methods and results to $n=2$. In particular, the Vandermonde matrix V is replaced by the confluent Vandermonde matrix U, which is defined as:

$$U:=[e^{ik\xi_j}\ \ ke^{i(k-1)\xi_j}]_{k=0,\dots,M}^{j=1,\dots,s}.$$

Under the partial clustering assumptions, we prove a sharp lower bound for the smallest singular value of U in the super-resolution regime, and show that it scales like $SRF^{2\ell-1}$. We also obtain sharp minimax bounds of order $SRF^{4\ell-1}\varepsilon$ for the problem of sparse superresolution on a grid.

Joint work with: Dmitry Batenkov.

References

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