Super-resolution on compact manifolds

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Consider the problem of recovering a Dirac spike train $F(x) = \sum_{k=1}^{K} a_k \delta_{x_k}$, where each $x_k \in (\mathbb{R}/2\pi\mathbb{Z})$ is an unknown point on a 1D circle, from its low-frequency trigonometric moments $\widehat{F}(n) = \sum_{k=1}^{K} a_k e^{jx_k n}$, for $|n| \le N$. This is the well-known question of sparse super-resolution, providing a popular model for several problems in computational mathematics and engineering, including spectral estimation, direction of arrival, imaging of point sources, sampling of signals with finite rate of innovation below the Nyquist limit, among others. Over the years, several extensions and generalizations of the above problem have been developed, including additional parametric models, and additional domains such as the torus or the sphere. On the other hand, allowing for high-order derivatives of Diracs enables to tackle problems such as recovery of polygons from moments, and high-accuracy recovery of piecewise-smooth functions [1].

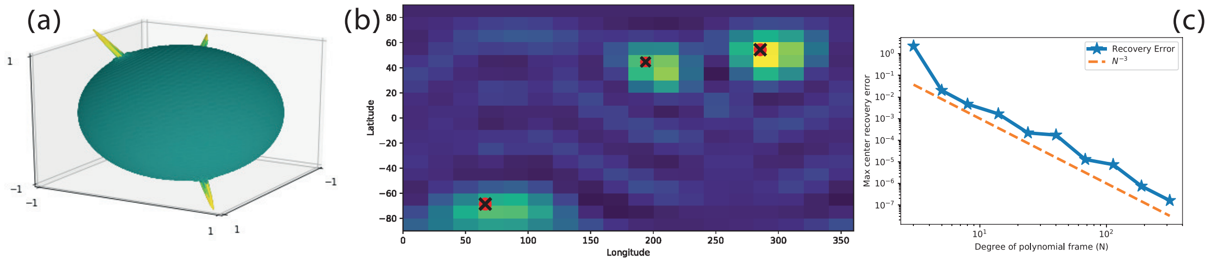

Figure 1: Generalized super-resolution on the sphere. (a) The original signal with $R = 2$ and $K = 3$. (b) An example reconstruction with $N = 8$ and additive error of magnitude $O(N^{-2})$. The red circles represent the true $\{x_k\}$ while the black crosses represent the reconstructed locations. The background is the original low-resolution data used in the algorithm. (c) Decay of the error is super-linear: $|\tilde{x_k} - x_k| \sim N^{-R-1}$.

Let $\mathcal{M}$ be a homogeneous compact Riemannian manifold (without boundary), and denote by $\{\lambda_n, \varphi_n\}_{n=0}^{\infty}$ the spectral decomposition of the manifold Laplacian, where $0 = \lambda_0^2 < \lambda_1^2 \le \cdots \uparrow +\infty$ are repeated according to multiplicity, while the real-valued orthonormal eigenfunctions $\{\varphi_n\}_{n=0}^{\infty}$ constitute a basis for the Hilbert space $L^2(\mathcal{M})$. In this work we consider the problem of super-resolution recovery of signals F which are modelled as sparse linear combinations $F \sim \sum_{k=1}^{K} \sum_{r=0}^{R} a_{k,r} G_r(x; x_k)$, where the "Bernoulli spline" G_r is the Green's function of the fractional Laplacian operator $(I + \Delta)^{r/2}$, from their low-frequency measurements $\widehat{F}_n := \langle F, \varphi_n \rangle_{\mathcal{M}}, \quad \lambda_n \leqslant N$. We build a constructive method to recover the model parameters by a hybrid two-stage approach. Assuming that the centers $\{x_k\}$ are separated by $\gtrapprox \frac{1}{N}$, we first apply a "sharpening" filter in the spectral domain, constructed using a localized Paley-Wiener type theory developed in [2] – obtaining an approximate locations of the centers $\{x_k\}$ and the amplitudes $\{a_{k,0}\}$. Subsequently, we fine-tune the initial estimates by a nonlinear least squares fit, and analyze the stability of this optimization problem. Our numerical experiments in Figure 1 for the 2D sphere with noisy data show super-linear convergence for recovering $\{x_k\}$ as $N \to \infty$. Our reconstruction procedure can in principle be used to localize supports of measures supported on curves, or submanifolds, of $\mathcal{M}$, and even on arbitrary metric measure spaces.

Joint work with: H.Mhaskar.

References

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[2] H. N. Mhaskar. A generalized diffusion frame for parsimonious representation of functions on data defined manifolds. *Neural Networks*, 24(4):345–359, 2011.