The uniform sparse FFT with application to PDEs with random coefficients

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Parametric operator equations have gained significant attention in recent years. In particular, partial differential equations with random coefficients play an important role in the study of uncertainty quantification. Therefore, the numerical solution of these equations and how to compute them in an efficient and reliable way has become more and more important. The usual example is the following model problem, describing the diffusion characteristics of inhomogeneous materials and therefore being called diffusion equation:

Let the domains $D \subseteq \mathbb{R}^{d_{\boldsymbol{x}}}$, typically with $d_{\boldsymbol{x}} = 1, 2$ or 3, and $D_{\boldsymbol{y}} \subseteq \mathbb{R}^d$ as well as the random coefficient $a : D \times D_{\boldsymbol{y}} \to \mathbb{R}$ and the right-hand side $f : D \to \mathbb{R}$ be given. Then, find $u : D \times D_{\boldsymbol{y}} \to \mathbb{R}$, such that for all $\boldsymbol{y} \in D_{\boldsymbol{y}}$ there holds

$$\begin{aligned} -\nabla \cdot (a(\boldsymbol{x}, \boldsymbol{y}) \nabla u(\boldsymbol{x}, \boldsymbol{y})) &= f(\boldsymbol{x}) && \boldsymbol{x} \in D,\ \boldsymbol{y} \in D_{\boldsymbol{y}} \\ u(\boldsymbol{x}, \boldsymbol{y}) &= 0 && \boldsymbol{x} \in \partial D,\ \boldsymbol{y} \in D_{\boldsymbol{y}}. \end{aligned}$$

The differential operator ∇ is always used w.r.t. the spatial variable $\boldsymbol{x}$. The one-dimensional components of $\boldsymbol{y}$ are assumed to be i.i.d. with a prescribed distribution and the dimension d of the random vector $\boldsymbol{y}$ is typically very large.

We present an efficient, non-intrusive algorithm to solve such differential equations, our so-called *uniform sparse FFT* or short *usFFT*. The aim is the approximation of the solution u on a discretization $\{\boldsymbol{x}_g \in D, g = 1, \dots, G\}, G < \infty$, of the spatial domain D, e.g., a finite element mesh. We receive an approximation

$$u_{\boldsymbol{x}_g}(\cdot) \coloneqq u(\boldsymbol{x}_g, \cdot) \approx \sum_{\boldsymbol{k} \in \mathrm{I}} c_{\boldsymbol{k}}^{\mathtt{usFFT}}(u_{\boldsymbol{x}_g})\, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \cdot}$$

for each node $\boldsymbol{x}_g$, where the frequency set I is a priori unknown. Our dimension-incremental approach realizes this by detecting a good frequency set I adaptively as well as computing approximations $c_{\boldsymbol{k}}^{\mathtt{usFFT}}(u_{\boldsymbol{x}_g}), \boldsymbol{k} \in \mathrm{I}$, of the corresponding Fourier coefficients for all nodes $\boldsymbol{x}_g$, $g = 1, \dots, G$, simultaneously. In particular, the usFFT does not need any a-priori information except for a suitable search domain $\Gamma \supset \mathrm{I}$, which can easily be chosen large enough without disturbing the algorithm. Further, the detected frequency set I is independent of g and therefore provides a good approximation basis for all $\boldsymbol{x}_g$, $g = 1, \cdots, G$, uniformly. Also, the usFFT only needs samples of the solution u at multiple points $\boldsymbol{y}$, which can be computed by any common PDE solver for each fixed $\boldsymbol{y}$. Hence, the method can be adapted to other types of differential equations, boundary conditions or domains easily by swaping this PDE solver with a suitable one for the new setting.

We test our algorithm on some examples with different diffusion coefficients $a(\boldsymbol{x}, \boldsymbol{y})$ and uniformly or standard normally distributed random variables $\boldsymbol{y}$ up to dimension $d = 20$. The results are analysed using common error norms. Further, the detected frequency set I as well as the approximated Fourier coefficients $c_{\boldsymbol{k}}^{\mathtt{usFFT}}(u_{\boldsymbol{x}_g}), \boldsymbol{k} \in \mathrm{I}$, provide detailed information on the influence and the interactions of the random variables $\boldsymbol{y}$.

Joint work with: Lutz Kämmerer, Daniel Potts.

References

[1] L. Kämmerer, D. Potts and F. Taubert. The uniform sparse FFT with application to PDEs with random coefficients. *ArXiv e-prints*, arXiv:2109.04131 [math.NA], 2021.