# $C^1$ Simplex–Splines on Simplices in $\mathbb{R}^s$.

Jean-Louis Merrien
Univ Rennes, INSA Rennes, France,
jmerrien@insa-rennes.fr

Piecewise polynomials over triangles and tetrahedrons have applications in several branches ranging from finite element analysis, surfaces in computer aided design... The smoothness on tetrahedrons is obtained either by high degrees of polynomials or using smaller degrees when splitting the tetraehedron into smaller pieces.

Here we consider the Alfeld split [1] which generalizes the Clough–Tocher split [2] of a triangle. To describe it, let $\mathcal{T}_s := \langle\{\boldsymbol{p}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_{s+1}\}\rangle$ be a simplex in $\mathbb{R}^s$. Using the barycenter $\boldsymbol{p}_{\mathcal{T}_s} := \sum_{j=1}^{s+1} \boldsymbol{p}_j/(s+1)$, we can split $\mathcal{T}_s$ into $s+1$ **subsimplices** $\mathcal{T}_{s,j} := \langle\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{s+1}, \boldsymbol{p}_{\mathcal{T}} \setminus \{\boldsymbol{p}_j\}\rangle$, $j = 1\ldots, s+1$. On $\mathcal{T}_s$ we consider the linear space of $C^1$ piecewise polynomials of degree $d \in \mathbb{N}_0$

$$\mathbb{S}^1_{d,s} := \{f \in C^1(\mathcal{T}_s) : f_{|\mathcal{T}_{s,j}} \in \mathbb{P}_d(\mathbb{R}^s),\, j = 1\ldots, s+1\}.$$

We denote by $\triangle[\boldsymbol{i};\ell] : \mathbb{R}^s \to \mathbb{R}$ the simplex spline with multiple knots $\{\boldsymbol{p}_1^{[i_1]}, \ldots, \boldsymbol{p}_{s+1}^{[i_{s+1}]}, \boldsymbol{p}_{\mathcal{T}}^{[\ell]}\}$, where the multiplicity vector $\boldsymbol{i} = (i_1, \ldots, i_{s+1})$ has nonnegative integer components. Generalizing [3], we consider degrees $d = 2s-1$ and construct a basis for $\mathbb{S}^1_{2s-1,s}$ consisting of simplex-splines $\triangle[\boldsymbol{i};\ell]$ for suitable $\boldsymbol{i}$ and $\ell$.

The first argument about the dimension was shown in [4], i.e. for $d \in \mathbb{N}_0$

$$\dim\left(\mathbb{S}^1_{d,s}\right) = \binom{d+s}{s} + s\binom{d-1}{s}.$$

Secondly, we focus on two types of elements of $\mathbb{S}^1_{2s-1,s}$.

- Type (0): the elements corresponding to Bernstein polynomials $\triangle(\boldsymbol{i};\ell) = B^{2s-1}_{\boldsymbol{i}-\boldsymbol{1}}$ with $\ell = 0$ and at least one $i_j$ is equal to one ,
- Type (1): the elements $\triangle(\boldsymbol{i};\ell)$ with $\ell > 0$, exactly one of the $i_j = 1$ and the others $i_k \geq 2$.

**Theorem:** The set of elements of type (0) and (1) is a basis of $\mathbb{S}^1_{2s-1,s}$
The theorem is completed by propositions on Marsden Identities and Domain Points.

**Joint work with:** Tom Lyche, Dept. of Mathematics, Univ. of Oslo, Norway, tom@math.uio.no.

## References

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