Infinite-dimensional STFT phase retrieval from lattice samples: uniqueness and stability

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The STFT phase retrieval problem arises as the result of trying to invert the mapping that sends a square-integrable function $f \in L^2(\mathbb{R})$ to spectrogram samples of the form

$$|V_g f(\mathcal{L})| := \{|V_g f(\ell)| : \ell \in \mathcal{L}\},$$

where $V_g f(x,\omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i\omega t}\,dt$ denotes the short-time Fourier transform of f with respect to a window function $g \in L^2(\mathbb{R})$ and $\mathcal{L} \subseteq \mathbb{R}^2$ is a sampling set in the time-frequency plane. An inversion of f from $|V_g f(\mathcal{L})|$ is only possible up to the ambiguity of a global phase factor. We say that f and h agree up to a global phase if there exists a constant $c \in \mathbb{C}, |c| = 1$, such that $f = ch$.

Regarded as a non-linear inverse problem, the investigation of uniqueness and stability of the STFT phase retrieval problem is of major importance. It is folklore in the phase retrieval community that under suitable assumptions on the window function g, every $f \in L^2(\mathbb{R})$ is determined up to a global phase from $|V_g f(\mathcal{L})|$ provided that $\mathcal{L}$ is a continuous domain such as an open set or the entire time-frequency plane. We are interested in the case where $\mathcal{L}$ is a separated set, most notably a lattice, i.e. $\mathcal{L} = L\mathbb{Z}^2$ for some $L \in \mathrm{GL}_2(\mathbb{R})$. First, we present a result which reveals a fundamental difference between the continuous and discrete case: there exists no window function $g \in L^2(\mathbb{R})$ and no lattice $\mathcal{L} \subset \mathbb{R}^2$ such that every $f \in L^2(\mathbb{R})$ is determined up to a global phase by $|V_g f(\mathcal{L})|$ [2, Theorem 1.2]. If the window function g is a Gaussian then the STFT phase retrieval problem is known as the Gabor phase retrieval problem. In this setting, uniqueness from lattice samples can be achieved by imposing a support condition: every $f \in L^4[-\frac{c}{2}, \frac{c}{2}]$ is determined up to a global phase by $|V_g f(\mathcal{L})|$ if g is a Gaussian and $\mathcal{L}$ is a rectangular lattice of the form $\mathcal{L} = \mathbb{Z} \times \frac{1}{2c}\mathbb{Z}$ [1, Theorem 3.1]. Finally, we consider shift-invariant spaces $\mathcal{V}_\beta^p(h) = \{\sum_{k\in\mathbb{Z}} c_k h(\cdot - \beta k) : \{c_k\} \in \ell^p(\mathbb{Z})\}$ where $h \in L^p(\mathbb{R})$ is a generating function and $\beta > 0$ is a step-size. We will demonstrate that Gabor phase retrieval in Gaussian shift-invariant spaces from lattice samples is possible under a suitable assumption on the step-size β which allows the application of the ergodic theorem [1, Theorem 3.6]. In addition, we highlight that the shift-invariant setting allows the design of a reconstruction algorithm which recovers functions in a provably and stably manner [3].

Joint work with: Philipp Grohs.

References

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