A framework for optimal convex regularization for the recovery of low-dimensional models

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We consider the problem of recovering an element x_0 of a low-dimensional model $\Sigma \subset \mathbb{R}^n$ (e.g. Σ_k the set of k-sparse vectors) from under-determined linear measurements $y = Mx_0$ where M is a linear map. To perform recovery, we consider the minimization of a convex regularizer subject to a data-fit constraint

$$x^* = \arg\min_{Mx=y} R(x). \tag{1}$$

This minimization can be proven successful for sparse models and their generalizations (such as low rank models) with the right choice of measurement matrices M (e.g. random Gaussian matrices with enough measurements) [2, 4, 1, 3]. Given a model, we ask ourselves what is the "best" convex regularizer to perform its recovery. A framework to define the optimality of a convex regularizer for the recovery of a given low dimensional model was introduced in [6]. Based on explicit recovery guarantees of elements of Σ, it defines optimal regularizers as functions R^* that maximize a *compliance measure* $A_\Sigma(R)$ that quantifies the recovery capabilities of elements of Σ by using minimization (1) with R:

$$R^* \in \arg\max_{R\in\mathcal{C}} A_\Sigma(R). \tag{2}$$

where $\mathcal{C}$ is a set of convex functions. It was shown that the ℓ^1-norm was an optimal atomic norm for the recovery of sparse models in minimal cases ($k = 3$) for compliance measures based on exact recovery guarantees and an optimal atomic norm in the general case for compliance measures based on best known recovery guarantees using the restricted isometry property [6, 7].

In this work (available as a full paper [5]), we build on these ideas and give elementary properties of the maximization of compliance measures. We show the optimality of the ℓ^1-norm and the nuclear norm for the recovery of sparse and low rank models respectively in the set of coercive continuous *convex* functions for compliances based on the restricted isometry property. Finally, we *construct* near-optimal regularizers for sparsity in levels models within the set of ℓ^1-norms weighed by levels. This result is a first example of explicit construction of optimal regularizers beyond classical sparsity models.

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